

CLASSIFICATION OF $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ -MANIFOLDS

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ABSTRACT. Let M be an analytic complete finite volume pseudo-Riemannian manifold and $\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})$ a connected semisimple Lie group such that its Lie algebra is $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$. We characterize the structure of the manifold M assuming that the Lie group $\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})$ acts isometrically on M and that its dimension satisfies $3 + n(2n+1) < \dim(M) \leq (n+1)(2n+3)$.

INTRODUCTION

Let G be a connected non-compact simple Lie group acting isometrically on a connected analytic manifold M with a pseudo-Riemannian metric of finite volume. It has been conjectured that such actions are rigid, in the sense that restrict the possibilities for M . Such conjecture is consequence of the program proposed by Robert Zimmer (see **). A principal belief is that such action together with other non-trivial assumptions imply that M is, basically, the double coset of a semisimple Lie group H . Specifically, we have a homomorphism $G \rightarrow H$, the existence of a compact subgroup $K \subset H$, centralizing the image of G , and a lattice $\Gamma \subset H$ such that M is isometric to $\Gamma \backslash H/K$.

Some results have been found in this subject, for example the actions of the Lie groups $SO(p, q)$ and $\widetilde{U}(p, q)$, where p and q are non-negative integer numbers (see [8],[9]). Note that in the latter case that the Lie group $\widetilde{U}(p, q)$ is not simple, therefore, there is evidence to suppose that the previous conjecture can be true with other Lie groups not necessarily simple.

In such context we present this research, here we analyze the isometric action of the semisimple Lie group $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ on a connected pseudo-Riemannian manifold of finite volume, assuming that both the action and the manifold are analytic.

In this paper, for any connected manifold N we denote by \widetilde{N} the simple connected universal covering of N . Let G_i be a non-compact, connected simple Lie group with Lie algebra \mathfrak{g}_i , $i = 1, 2$. In this case $G := G_1 \times G_2$ is a semisimple Lie group without compact factors with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let M be a connected, finite-volume, pseudo Riemannian manifold which admits an analytic and isometric G -action with a dense orbit where no factor of G acts trivially. As in [8], we prove that for M a weakly irreducible and complete manifold there is a lower bound of its dimension given by the dimension of the semisimple Lie group and the properties of the representations of its Lie algebra. In other words

$$\dim(M) \geq \dim(G) + m_0(\mathfrak{g}_1, \mathfrak{g}_2)$$

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where $m_0(\mathfrak{g}_1, \mathfrak{g}_2)$ denotes the dimension of the smallest non-trivial representation of both Lie algebras, \mathfrak{g}_1 and \mathfrak{g}_2 , preserving a non-degenerate symmetric bilinear form.

As a consequence of our research we have the following theorem whose main result says that such action, together with other conditions on the manifold and the action, imply that M is isomorphic, up to a finite covering, to a quotient map of a simple Lie group over a lattice.

Theorem A. *We assume the semisimple Lie group $G = (\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$, for $n \geq 3$, acts isometrically with a dense orbit on a connected, finite-volume, complete, pseudo-Riemannian manifold M , where no factor of G acts trivially. Assume that M and the G -action on M are both analytic. If M is weakly irreducible and satisfies that $\dim(M) = (n+1)(2n+3)$, then there exist:*

- a lattice $\Gamma \subset \widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$, and
- an analytic finite covering map $\tau : \widetilde{\mathrm{Sp}}(n+1, \mathbb{R})/\Gamma \rightarrow M$,

such that τ is $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ -equivariant. We can also rescale the metric on M along the $\widetilde{\mathrm{Sp}}(n, \mathbb{R})$ and $\widetilde{\mathrm{Sp}}(1, \mathbb{R})$ -orbits and the normal bundle to the $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ -orbits, such that τ is a local isometry for the metric on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ given by the Killing form of its Lie algebra.

The proof of our principal result is based in the tools developed by Gromov and Zimmer through the study of the properties of representation of the Killing vector fields that centralize the action of the semisimple group. One of our principal tools is Proposition 1.2, which is a generalization of a similar result in [8] and [10]. Such proposition shows the existence of a Lie algebra $\mathfrak{g}(x)$, isomorphic to $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$, of Killing vector fields vanishing in a point $x \in \widetilde{M}$. Such Lie algebra $\mathfrak{g}(x)$ induces a structure of \mathfrak{g} -module on $T_x \widetilde{M}$ which allows the use of representation theory to analyze the normal bundle to the foliation generated by the orbits of the action. The \mathfrak{g} -module structure of $T_x \widetilde{M}$ is closely related to a structure of \mathfrak{g} -module of \mathcal{H} , the set of Killing vector fields that centralize the action of the group G . Such structure gives us more tools to understand the properties of \mathcal{H} , which instead gives place to the action of another Lie group on M . The proof of the existence of the centralizer \mathcal{H} of the action can be found in Section 1. In Section 2 we analyze the properties of \mathcal{H} and its relation with the tangent space at some point in M . The new action induced by the centralizer is an important tool for the proof of Theorem A, which can be found in Section 3. Meanwhile, in Appendix A we have results about the representations of $\widetilde{\mathrm{Sp}}(n, \mathbb{R})$ and its Lie algebra $\mathfrak{sp}(n, \mathbb{R})$, which are used in the previous sections.

1. FIRST RESULTS

Let G be a semisimple Lie group as in the introduction. We assume that G acts isometrically with a dense orbit on a connected, finite-volume, pseudo-Riemannian manifold M where no factor of G acts trivially. Hence, the G -action is locally free (see [11, Theorem 4.17]) and its orbits define a foliation that we denote by \mathcal{F} . We also denote by \mathcal{F}_1 (resp. \mathcal{F}_2) the foliation defined by the G_1 -orbits (resp. G_2 -orbits). We consider that M and the G -action on M are both analytic.

For $X \in \mathfrak{g}$, we denote by X^* the vector field on the manifold M whose one-parameter group of diffeomorphism is given by $(\exp(tX))_t$ through the action on the manifold.

For any given pseudo-Riemannian manifold N , we will denote by $\mathrm{Kill}(N)$ the globally defined Killing vector fields of N . We denote by $\mathrm{Kill}_0(N, x)$ the Lie algebra of Killing vector fields that vanish at the given point x . For a vector space W with a non-degenerate symmetric bilinear form, we will denote by $\mathfrak{so}(W)$ the Lie algebra of linear maps on W that are skew-symmetric with respect to the bilinear form. The next result is an application of the Jacobi identity.

Lemma 1.1. *Let N be a pseudo-Riemannian manifold and $x \in N$. Then, the map $\lambda_x : \mathrm{Kill}_0(N, x) \rightarrow \mathfrak{so}(T_x N)$ given by $\lambda_x(Z)(v) = [Z, V]_x$, where V is any vector field such that $V_x = v$, is a well-defined homomorphism of Lie algebras.*

An immediate consequence of the previous result is its use in the proof of the following proposition which is a generalization of Proposition 1.2 in [8].

Proposition 1.2. *Assume that G acts isometrically with a dense orbit on a connected, finite-volume, pseudo-Riemannian manifold M , where no factor of G acts trivially. Consider the \widetilde{G} -action on \widetilde{M} lifted from the G -action on M . Assume that M and the G -action on M are both analytic. Then, there exists a conull subset $S \subset \widetilde{M}$ such that for every $x \in S$ the following properties are satisfied:*

- (1) *There is a homomorphism $\rho : \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathrm{Kill}(\widetilde{M})$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.*
- (2) *$\mathfrak{g}(x) \subset \mathrm{Kill}_0(\widetilde{M}, x)$, i.e. every element of $\mathfrak{g}(x)$ vanishes at x .*
- (3) *For every $X, Y \in \mathfrak{g}$ we have:*

$$[\rho_x(X), Y^*] = [X, Y]^* = -[X^*, Y^*].$$

In particular, the elements in $\mathfrak{g}(x)$ and their corresponding local flows preserve \mathcal{F} , \mathcal{F}_1 , \mathcal{F}_2 and $T\mathcal{F}^\perp$.

- (4) *The homomorphism of Lie algebras $\lambda_x \circ \rho_x : \mathfrak{g} \rightarrow \mathfrak{so}(T_x \widetilde{M})$ induces a \mathfrak{g} -module structure on $T_x \widetilde{M}$ for which the subspaces $T_x \mathcal{F}$, $T_x \mathcal{F}_1$, $T_x \mathcal{F}_2$ and $T_x \mathcal{F}^\perp$ are \mathfrak{g} -submodules.*
- (5) *For every $X_i \in \mathfrak{g}_i$, $i = 1, 2$, we have*

$$[\rho_x(X_i), X_j^*] = [X_i, X_j]^* = 0 \quad \text{with } i \neq j.$$

Proof. The proof is similar to the proof of Proposition 1.2 in [10]. We only note that the result: $\overline{\mathrm{Ad}(G)}^Z$ is the algebraic hull of $M \times GL(\mathfrak{g})$ for the product action, is also true for semisimple Lie groups without compact factors (see [13, Example 3.15]). \square

Let $x \in S$ and $u \in T_x \mathcal{F}_1 \cap T_x \mathcal{F}_2$, then there exists $X_i \in \mathfrak{g}_i$, for $i = 1, 2$, such that $(X_1^*)_x = (X_2^*)_x = u$. Let $Y_j \in \mathfrak{g}_j$ be, for $j = 1, 2$, by Lemma 1.1 and Proposition 1.2(3) we have that

$$\lambda_x(\rho_x(Y_1))(u) = [\rho_x(Y_1), X_2^*]_x = [Y_1, X_2]^*_x = 0,$$

and

$$\lambda_x(\rho_x(Y_2))(u) = [\rho_x(Y_2), X_1^*]_x = [Y_2, X_1]^*_x = 0,$$

which imply that $u = 0$. Since $x \in S$ is arbitrary we conclude that $T_x \mathcal{F} = T_x \mathcal{F}_1 \oplus T_x \mathcal{F}_2$, for every $x \in S$.

With the above setup, assume that the G_i -orbits are non-degenerate with respect to the ambient pseudo-Riemannian metric. In particular, the \widetilde{G}_i -orbits on \widetilde{M} are non-degenerate as well and we have a direct sum decomposition $T\widetilde{M} = T\mathcal{F}_i \oplus T\mathcal{F}_i^\perp$. Recall the differential form ω^i (see [10]) given, at every point $x \in \widetilde{M}$, by the composition of the natural projection $T_x\widetilde{M} \rightarrow T_x\mathcal{F}_i$ and the natural isomorphism $T_x\mathcal{F}_i \cong \mathfrak{g}_i$. We also recall the \mathfrak{g}_i -valued 2-form given by $\Omega^i = d\omega^i|_{\wedge^2 T\mathcal{F}_i^\perp}$, for $i = 1, 2$.

Lemma 1.3 ([10, Lemma 2.5]). *Let G , M , and S be as in Proposition 1.2. If we assume that the G_i -orbits are non-degenerate, for $i = 1, 2$, then:*

- (1) *For every $x \in S$, the maps $\omega_x^i : T_x\widetilde{M} \rightarrow \mathfrak{g}_i$ and $\Omega_x^i : \wedge^2 T_x\mathcal{F}_i^\perp \rightarrow \mathfrak{g}_i$ are \mathfrak{g}_i -homomorphism, for the \mathfrak{g}_i -module structures from Proposition 1.2.*
- (2) *The normal bundle $T\mathcal{F}_i^\perp$ is integrable if and only if $\Omega^i = 0$.*

Proof. The proof is similar as that of Lemma 2.5 in [10], where the simplicity (or semisimplicity) of the group does not play a role. \square

Next, we relate the metric of $T\mathcal{F}$ coming from M to suitable metrics on G .

Lemma 1.4 ([10, Lemma 2.6]). *Suppose that the G -action on M has a dense orbit and preserves a finite-volume pseudo-Riemannian metric. Then, for every $x \in M$ and with respect to the natural isomorphism $T_x\mathcal{F} \cong \mathfrak{g}$, the metric of M restricted to $T_x\mathcal{F}$ defines an $\text{Ad}(G)$ -invariant symmetric bilinear form on \mathfrak{g} independent of the point x .*

Proof. See the proof at Lemma 2.6 in [10]. \square

We assume, from now on, that $\dim(G_2) < \dim(G_1)$. In this case we have that the non-degeneracy of orbits is ensured for low-dimensional manifolds by the next result, which is similar to [10, Lemma 2.7].

Lemma 1.5. *Assume that $G = G_1 \times G_2$ acts isometrically and with a dense orbit on a connected finite-volume pseudo-Riemannian manifold M . If $\dim(M) < 2\dim(G_1)$ and if the G_2 -orbits are non-degenerate, then the bundles $T\mathcal{F}_1$, $T\mathcal{F}$ and $T\mathcal{F}^\perp$ have fibers that are non-degenerate with respect to the metric on M .*

Proof. By Lemma 1.4, for every $x \in M$, the metric h (on M) restricted to $T_x\mathcal{F}$ corresponds to an $\text{Ad}(G)$ -invariant form in \mathfrak{g} . The Kernel of such form is \mathfrak{g} -invariant, therefore, since $T_x\mathcal{F}_i \simeq \mathfrak{g}_i$, $\ker(h_x)$ is either 0, $T_x\mathcal{F}_1$, $T_x\mathcal{F}_2$ or $T_x\mathcal{F}$.

Assume that $T_x\mathcal{F}_1 \subseteq \ker(h_x)$ for some $x \in M$. Then, $T_x\mathcal{F}_1$ lies in the null cone of $T_x\mathcal{F}$ for the metric h_x . Hence, for the signature of M , which we denote as (m, n) , we have that $\dim(G_1) = \dim(T_x\mathcal{F}_1) \leq \min(m, n)$. This implies that $2\dim(G_1) \leq m + n = \dim(M)$, which is impossible.

On the other hand, by hypothesis $\ker(h_x) \neq T_x\mathcal{F}_2$. Therefore and the previous paragraphs we have the desired result. \square

Remark 1.6. Let us choose and fix an element $x \in \mathrm{S}$. Let $X_i \in \mathfrak{g}_i$ be, for $i = 1, 2$, if $Z_1 \in \mathfrak{g}_1$ then, by Proposition 1.2(2)-(3), $\rho_x(Z_1) \in \mathrm{Kill}_0(\widetilde{M}, x)$ and

$$\begin{aligned} \langle (X_1)_x^*, (X_2)_x^* \rangle_x &= \langle \rho_x(Z_1) \cdot (X_1)_x^*, \rho_x(Z_1) \cdot (X_2)_x^* \rangle_x \\ &= \langle [Z_1, X_1]_x^*, [Z_1, X_2]_x^* \rangle_x \\ &= \langle [Z_1, X_1]_x^*, 0_x^* \rangle_x \\ &= \langle [Z_1, X_1]_x^*, 0 \rangle_x \\ &= 0. \end{aligned}$$

The previous computation and Lemma 1.5 imply that $T_x \mathcal{F}$ is an orthogonal direct sum of $T_x \mathcal{F}_1$ and $T_x \mathcal{F}_2$. In particular, $T_x \mathcal{F}_1^\perp$ is the orthogonal direct sum of $T_x \mathcal{F}_2$ and $T_x \mathcal{F}^\perp$.

If the G -orbits are non-degenerate and the normal bundle to such orbits is integrable, then the universal covering space can be split.

Proposition 1.7. *Assume G_i (resp. G) acts isometrically on a connected, complete, finite-volume, pseudo-Riemannian manifold M . If the tangent bundle to the orbits $T\mathcal{F}_i$ (resp. $T\mathcal{F}$) has non-degenerate fibers and the bundle $T\mathcal{F}_i^\perp$ (resp. $T\mathcal{F}^\perp$) is integrable, then there is an isometric covering map $\widetilde{G}_i \times N \rightarrow M$ (resp. $\widetilde{G} \times N \rightarrow M$) where the domain has the product metric for a bi-invariant metric on \widetilde{G}_i (resp. \widetilde{G}) and with N a complete pseudo-Riemannian manifold, for $i = 1, 2$.*

Recall, from the proof of Lemma 1.3, that for $X_1 \in \mathfrak{g}_1$ and if $u, v \in T_x \mathcal{F}_1^\perp$ with U, V sections of $T\mathcal{F}_1^\perp$ extending them, we have that

$$(1.1) \quad X_1 \cdot \Omega_x^1(u \wedge v) = -\omega_x^1([\rho_x(X_1), U], V)_x - \omega_x^1([U, \rho_x(X_1)], V)_x,$$

in a similar way, we have the same result for the homomorphism Ω_x^2 .

Lemma 1.8. *Let G , M and S be as in Proposition 1.2. Assume that M is complete and weakly irreducible. Then, for almost every $x \in \mathrm{S}$ we have that $T_x \mathcal{F}^\perp$ is a non-trivial \mathfrak{g}_i -module, $i = 1, 2$.*

Proof. Since \widetilde{M} is a weakly irreducible manifold, by Proposition 1.7 and Lemma 1.3(2) we have that $\Omega^1 \neq 0$, therefore, since the 2-form Ω^1 is clearly analytic, it vanishes on a proper analytic submanifold subset of \widetilde{M} of measure zero. Hence, $\Omega_x^1 \neq 0$ for almost every $x \in \mathrm{S}$. Let $x \in \mathrm{S}$ be an arbitrary but fixed element such that $\Omega_x^1 \neq 0$. Lemma 1.3(1) implies that the map $\Omega_x^1 : T_x \mathcal{F}_1^\perp \rightarrow \mathfrak{g}_1$ is a non-trivial map.

Let $u, v \in T_x \mathcal{F}_1^\perp$, by Remark 1.6, there are $u_2, v_2 \in T_x \mathcal{F}_2$ and $\hat{u}, \hat{v} \in T_x \mathcal{F}^\perp$ such that $u = u_2 + \hat{u}$ and $v = v_2 + \hat{v}$. Let U, V be sections of $T\mathcal{F}^\perp$ such that $U_x = \hat{u}$ and $V_x = \hat{v}$, in the same way, let $X_2, Y_2 \in \mathfrak{g}_2$ be such that $(X_2)_x^* = u_2$ and $(Y_2)_x^* = v_2$.

Now, let $X_1 \in \mathfrak{g}_1$, by (1.1), we have

$$\begin{aligned}
X_1 \cdot \Omega_x^1(u \wedge v) &= X_1 \cdot \Omega_x^1((u_2 + \hat{u}) \wedge (v_2 + \hat{v})) \\
&= X_1 \cdot \Omega_x^1(u_2 \wedge v_2) + X_1 \cdot \Omega_x^1(u_2 \wedge \hat{v}) + \\
&\quad X_1 \cdot \Omega_x^1(\hat{u} \wedge v_2) + X_1 \cdot \Omega_x^1(\hat{u} \wedge \hat{v}) \\
&= -\omega_x^1([\rho_x(X_1), X_2^*], Y_2^*]_x) - \omega_x^1([X_2^*, [\rho_x(X_1), Y_2^*]]_x) + \\
&\quad -\omega_x^1([\rho_x(X_1), X_2^*], U]_x) - \omega_x^1([X_2^*, [\rho_x(X_1), U]]_x) + \\
&\quad -\omega_x^1([\rho_x(X_1), U], Y_2^*]_x) - \omega_x^1([U, [\rho_x(X_1), Y_2^*]]_x) + \\
&\quad -\omega_x^1([\rho_x(X_1), U], V]_x) - \omega_x^1([U, [\rho_x(X_1), V]]_x) \\
&= -\omega_x^1([0, Y_2^*]_x) - \omega_x^1([X_2^*, 0]_x) + \\
&\quad -\omega_x^1([0, U]_x) - \omega_x^1([X_2^*, 0]_x) + \\
&\quad -\omega_x^1([0, Y_2^*]_x) - \omega_x^1([U, 0]_x) + \\
&\quad -\omega_x^1([0, V]_x) - \omega_x^1([U, 0]_x) \\
&= 0.
\end{aligned}$$

As $X_1 \in \mathfrak{g}_1$ was arbitrary, it follows that $\Omega_x^1(T_x \mathcal{F}_2 \wedge T_x \mathcal{F}^\perp) = 0$. On the other hand, because $\Omega_x^1 \neq 0$, we have that $\Omega_x^1(\wedge^2 T_x \mathcal{F}^\perp) \neq 0$, therefore, we have that $T_x \mathcal{F}^\perp$ is not a trivial \mathfrak{g}_1 -module.

In a similar way, to the previous steps, we can prove that $T_x \mathcal{F}^\perp$ is not a trivial \mathfrak{g}_2 -module. \square

For $i = 1, 2$, let $m(\mathfrak{g}_i)$ be the dimension of the smallest non-trivial representation of \mathfrak{g}_i that admits an invariant non-degenerate symmetric bilinear form. Since $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, we define $m(\mathfrak{g}) = m(\mathfrak{g}_1, \mathfrak{g}_2)$ the dimension of the smallest non-trivial representation of both \mathfrak{g}_1 and \mathfrak{g}_2 that admits an invariant non-degenerate symmetric bilinear form. If we assume that there is a non-trivial homomorphism $\mathfrak{g}_2 \rightarrow \mathfrak{g}_1$ then $m(\mathfrak{g}) \leq m(\mathfrak{g}_1)$, even more we have that $m(\mathfrak{g}) = m(\mathfrak{g}_1)$.

From now on we assume the existence of an injective homomorphism $\mathfrak{g}_2 \rightarrow \mathfrak{g}_1$. A consequence of the previous result is the obtention of a lower bound on the dimension of M .

Proposition 1.9. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, weakly irreducible, has finite-volume and admits an analytic isometric, non-transitive G -action with a dense orbit and such that no factor acts trivially. We also assume that the G_2 -orbits are non-degenerate. If $m(\mathfrak{g}) + \dim(G_2) \leq \dim(G_1)$ then*

$$\dim(M) \geq \dim(G) + m(\mathfrak{g}).$$

Proof. Suppose that $\dim(M) < \dim(G) + m(\mathfrak{g})$. Since $m(\mathfrak{g}) + \dim(G_2) \leq \dim(G_1)$ then $\dim(M) < 2 \dim(G_1)$ and, by Lemma 1.5, the bundle $T\mathcal{F}^\perp$ has non-degenerate fibers with dimension $< m(\mathfrak{g})$. Hence, Lemma 1.8 and the definition of $m(\mathfrak{g})$ imply that $T_x \mathcal{F}^\perp$ is a trivial \mathfrak{g}_1 -module for the structure defined by Proposition 1.2(4), hence Proposition 1.7 contradicts the irreducibility of M . \square

For a G -action as in Proposition 1.2, consider \widetilde{M} endowed with the \widetilde{G} -action obtained by lifting the G -action on M . With such setup, let us denote by \mathcal{H} the Lie subalgebra of $\text{Kill}(\widetilde{M})$ consisting of the fields that centralize the \widetilde{G} -action on \widetilde{M} .

Lemma 1.10. *Let S be as in Proposition 1.2. Then, for every $x \in S$ and for ρ_x as in Proposition 1.2, the map $\widehat{\rho}_x : \mathfrak{g} \rightarrow \mathrm{Kill}(\widetilde{M})$ given by:*

$$\widehat{\rho}(X) = \rho_x(X) + X^*$$

is an injective homomorphism of Lie algebras whose image $\mathcal{G}(x)$ lies in \mathcal{H} . In particular, $\widehat{\rho}_x$ induces on \mathcal{H} a \mathfrak{g} -module structure such that $\mathcal{G}(x)$ is a submodule isomorphic to \mathfrak{g} .

Proof. The identity in Proposition 1.2(3) implies that the image of $\widehat{\rho}_x$ lies in \mathcal{H} .

Let $X_i, Y_i \in \mathfrak{g}_i$ be, for $i = 1, 2$, then applying Proposition 1.2(3) we have

$$\begin{aligned} [\widehat{\rho}_x(X_1 + X_2), \widehat{\rho}_x(Y_1 + Y_2)] &= [\rho_x(X_1 + X_2) + (X_1 + X_2)^*, \\ &\quad \rho_x(Y_1 + Y_2) + (Y_1 + Y_2)^*] \\ &= [\rho_x(X_1) + X_1^* + \rho_x(X_2) + X_2^*, \\ &\quad \rho_x(Y_1) + Y_1^* + \rho_x(Y_2) + Y_2^*] \\ &= \sum_{i,j=1}^2 [\rho_x(X_i) + X_i^*, \rho_x(Y_j) + Y_j^*] \\ &= \sum_{i,j=1}^2 ([\rho_x(X_i), \rho_x(Y_j)] + [\rho_x(X_i), Y_j^*] + \\ &\quad [X_i^*, \rho_x(Y_j)] + [X_i^*, Y_j^*]) \\ &= \sum_{i,j=1}^2 (\rho_x([X_i, Y_j]) + [X_i, Y_j]^* + \\ &\quad [X_i, Y_j]^* - [X_i, Y_j]^*) \\ &= \sum_{i,j=1}^2 (\rho_x([X_i, Y_j]) + [X_i, Y_j]^*) \\ &= (\rho_x([X_1 + X_2, Y_1 + Y_2]) + [X_1 + X_2, Y_1 + Y_2]^*) \\ &= \widehat{\rho}_x([X_1 + X_2, Y_1 + Y_2]). \end{aligned}$$

If $X \in \mathfrak{g}$ is an element which satisfies that $\widehat{\rho}_x(X) = 0$ then $X^* = \rho_x(X) + X^* = 0$, which, by locally freeness, implies $X = 0$. \square

Following, we relate the \mathfrak{g} -module structure associated to \mathcal{H} and to $T_x \widetilde{M}$, respectively.

Lemma 1.11. *Let S be as in Proposition 1.2. Consider $T_x \widetilde{M}$ and \mathcal{H} endowed with the \mathfrak{g} -module structure given by Proposition 1.2 and Lemma 1.10, respectively. Then, for every $x \in S$, the evaluation map*

$$\mathrm{ev}_x : \mathcal{H} \rightarrow T_x \widetilde{M}, \quad Y \mapsto Y_x$$

is a homomorphism of \mathfrak{g} -modules that satisfies $\mathrm{ev}_x(\mathcal{G}(x)) = T_x \mathcal{F}$. Furthermore, for almost every $x \in S$ we have $\mathrm{ev}_x(\mathcal{H}) = T_x \widetilde{M}$.

Proof. For every $x \in S$, let $Y \in \mathcal{H}$ and $X \in \mathfrak{g}$ be given, then

$$\begin{aligned}
 \text{ev}_x(X \cdot Y) &= [\widehat{\rho}_x(X), Y]_x \\
 &= [\rho_x(X) + X^*, Y]_x \\
 &= [\rho_x(X), Y]_x + [X^*, Y]_x \\
 &= [\rho_x(X), Y]_x \\
 &= \rho_x(X) \cdot Y_x \\
 &= X \cdot \text{ev}_x(Y)
 \end{aligned}$$

where we have used the definition of \mathfrak{g} -module structures involved and properties of the map λ_x (Lemma 1.1). The last claim follows by an adaptation of the proof of Lemma 4.1 of [14] and Theorem 3.1 of [5], which establish the transitivity of \mathcal{H} on an open conull dense. \square

On a complete manifold every Lie algebra of Killing vector fields can be realized from an isometric right action, this is the result of the following Lemma which appears as Lemma 1.11 in [8].

Lemma 1.12 ([8, Lemma 1.11]). *Let N be a complete pseudo-Riemannian manifold and H a simply connected Lie group with Lie algebra \mathfrak{h} . If $\psi : \mathfrak{h} \rightarrow \text{Kill}(N)$ is a homomorphism of Lie algebras, then there exists an isometric right H -action $N \times H \rightarrow N$ such that $\psi(X) = X^*$, for every $X \in \mathfrak{h}$. Furthermore, if N is analytic, then the H -action is analytic as well.*

2. STRUCTURE OF THE CENTRALIZER

In this section we assume the case $G = (\widetilde{\text{Sp}}(n, \mathbb{R}) \times \widetilde{\text{Sp}}(1, \mathbb{R}))$ which acts analytical and isometrically on a connected, analytic, finite-volume, complete, pseudo-Riemannian manifold M with a dense orbit, such that no factors of G acts trivially. Therefore, the results of Section 1 can apply to this case. We also assume that $\dim(M) \leq \dim(\widetilde{\text{Sp}}(n, \mathbb{R}) \times \widetilde{\text{Sp}}(1, \mathbb{R})) + 4n = 2n^2 + 5n + 3$, for $n \geq 3$.

Given the assumptions in the previous paragraph, by Lemma 1.5, we have the direct sum $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$. Here, we also assume that the manifold M is weakly irreducible.

Lemma 2.1. *Let $x \in S$. Consider $T_x\mathcal{F}^\perp$ endowed with the $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module structure given by Proposition 1.2(4). Then, for almost every $x \in S$, the $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module $T_x\mathcal{F}^\perp$ is isomorphic to $\mathbb{R}^{2n, 2n}$. In particular, $\mathfrak{so}(T_x\mathcal{F}^\perp)$ is isomorphic to $\mathfrak{so}(2n, 2n)$ as a Lie algebra and as an $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module.*

Proof. Since we are assuming that M is weakly irreducible, by Lemma 1.8, we have that for almost every $x \in S$, $T_x\mathcal{F}^\perp$ is a non-trivial $\mathfrak{sp}(n, \mathbb{R})$ -module (respectively $\mathfrak{sp}(1, \mathbb{R})$ -module), therefore it is a non-trivial $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module.

By Proposition 1.2(4) and Lemma 1.5 we have that the map $\lambda_x \circ \rho_x$ induces a non-trivial homomorphism of Lie algebras from $\mathfrak{sp}(n, \mathbb{R})$ (resp. $\mathfrak{sp}(1, \mathbb{R})$) into $\mathfrak{so}(T_x\mathcal{F}^\perp)$. Since $\mathfrak{sp}(n, \mathbb{R})$ (resp. $\mathfrak{sp}(1, \mathbb{R})$) is a simple Lie algebra we have that such homomorphism is injective.

Let $X \in \mathfrak{sp}(n, \mathbb{R})$ and $Y \in \mathfrak{sp}(1, \mathbb{R})$ be, if $u \in T_x \mathcal{F}^\perp$ we chose U a vector field on \widetilde{M} such that $U_x = u$, then

$$\begin{aligned}
 ((\lambda_x \circ \rho_x)(X) \circ (\lambda_x \circ \rho_x)(Y))(u) &= ((\lambda_x \circ \rho_x)(X) \circ (\lambda_x \circ \rho_x)(Y))(U_x) \\
 &= (\lambda_x \circ \rho_x)(X)(\lambda_x(\rho_x(Y)))(U_x) \\
 &= (\lambda_x \circ \rho_x)(X)([\rho_x(Y), U]_x) \\
 &= (\lambda_x(\rho_x(X)))([\rho_x(Y), U]_x) \\
 &= [\rho_x(X), [\rho_x(Y), U]]_x \\
 &= [\rho_x(Y), [\rho_x(X), U]]_x + [[\rho_x(X), \rho_x(Y)], U]_x \\
 &= [\rho_x(Y), [\rho_x(X), U]]_x + [\rho_x([X, Y]), U]_x \\
 &= [\rho_x(Y), [\rho_x(X), U]]_x \\
 &= ((\lambda_x \circ \rho_x)(Y) \circ (\lambda_x \circ \rho_x)(X))(u)
 \end{aligned}$$

therefore, we have that $(\lambda_x \circ \rho_x)(\mathfrak{sp}(n, \mathbb{R}))$ and $(\lambda_x \circ \rho_x)(\mathfrak{sp}(1, \mathbb{R}))$ commute each other in $\mathfrak{so}(T_x \mathcal{F}^\perp)$. Hence, the map $\lambda_x \circ \rho_x$ induces an injective homomorphism of Lie algebras from $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ into $\mathfrak{so}(T_x \mathcal{F}^\perp)$.

Since $T_x \mathcal{F}^\perp$ is a non-trivial $\mathfrak{sp}(n, \mathbb{R})$ -module preserving a non-degenerate symmetric bilinear form then, by Lemma A.7, we have that $T_x \mathcal{F}^\perp$ is isomorphic to $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ as $\mathfrak{sp}(n, \mathbb{R})$ -module. Let $\{e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots, e_{n+n}\}$ be the canonical base of \mathbb{R}^{2n} . Let $i, j \in \{1, 2, \dots, n\}$ be such that $i \neq j$, by the representation of $\mathfrak{sp}(n, \mathbb{R})$ on \mathbb{R}^{2n} we can find $A_{i,j} \in \mathfrak{sp}(n, \mathbb{R})$ such that $A_{i,j}(e_i) = e_i$ and $A_{i,j}(e_j) = e_j$, therefore if $\langle \cdot, \cdot \rangle$ is a non-degenerate symmetric bilinear form on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ preserved by $\mathfrak{sp}(n, \mathbb{R})$ then

$$0 = \langle A_{i,j}(e_i), e_j \rangle + \langle e_i, A_{i,j}(e_j) \rangle = \langle e_i, e_j \rangle + \langle e_i, e_j \rangle = 2\langle e_i, e_j \rangle.$$

Hence, a subspace of dimension $2n$ is contained in the nullcone of $\langle \cdot, \cdot \rangle$ therefore we have that the signature of $\langle \cdot, \cdot \rangle$ is $(2n, 2n)$. \square

The results in the previous lemma has an immediate consequence in the proof of the following lemma.

Lemma 2.2. *Let S be as in Proposition 1.2. Then, for almost every $x \in S$ and for the $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module structure on \mathcal{H} from Lemma 1.10 there is a decomposition into $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -submodules $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x)$ such that*

- (1) $\mathcal{G}(x) = \widehat{\rho}_x(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ is a Lie subalgebra of \mathcal{H} isomorphic to $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ and $\text{ev}_x(\mathcal{G}(x)) = T_x \mathcal{F}$.
- (2) $\mathcal{H}_0(x) = \ker(\text{ev}_x)$, is a Lie subalgebra of \mathcal{H} isomorphic to a subalgebra of $\mathfrak{so}(2n, 2n)$. Even more, such isomorphism is an isomorphism of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules.
- (3) $\text{ev}_x(\mathcal{W}(x)) = T_x \mathcal{F}^\perp$ and is isomorphic to $\mathbb{R}^{(2n, 2n)}$ as $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module.

Here, the evaluation map ev_x defines an isomorphism of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules $\mathcal{G}(x) \oplus \mathcal{W}(x) \rightarrow T_x \widetilde{M} = T_x \mathcal{F} \oplus T_x \mathcal{F}^\perp$ preserving the summands in that order.

Proof. Let $x \in S$ which satisfies Lemma 1.11 and Lemma 2.1. Recall, by Lemma 1.10, that $\mathcal{G}(x) = \widehat{\rho}_x(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ is a Lie subalgebra contained in \mathcal{H} and isomorphic to $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$.

Define $\mathcal{H}_0(x) = \ker(\text{ev}_x)$. By Lemma 1.11, we have that $\mathcal{H}_0(x)$ is an $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module of \mathcal{H} . Since, $\mathcal{H}_0(x) = \mathcal{H}(x) \cap \text{Kill}_0(\widetilde{M}, x)$ it follows that it is a subalgebra.

Let $Z \in \mathcal{G}(x) \cap \mathcal{H}_0(x)$ be, then there is $Y \in \mathfrak{g}$ such that $Z = \widehat{\rho}_x(Y) = \rho_x(Y) + Y^*$. The condition $Z \in \mathcal{H}_0(x)$ implies $0 = Z_x$. That is, $Y_x^* = (\rho_x(Y) + Y^*)_x = 0$, hence, $Y = 0$ and we have that $\mathcal{G}(x) \cap \mathcal{H}_0(x) = \{0\}$. Therefore, by Lemma 1.11, there is a subspace $\mathcal{W}_0(x)$ complementary to $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ in \mathcal{H} .

Since we have an isomorphism from $\mathcal{G}(x) \oplus \mathcal{W}_0(x)$ onto $T_x \widetilde{M}$ via the evaluation map, we choose $\mathcal{W}(x)$ as the inverse image of $T_x \mathcal{F}^\perp$ under this isomorphism. Therefore, we obtain the desired composition of \mathcal{H} into $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules.

Let $\text{Kill}_0(\widetilde{M}, x, \mathcal{F})$ be the Lie algebra of Killing vector fields on \widetilde{M} which preserves the foliation \mathcal{F} and vanish at $x \in \widetilde{M}$. Note that every vector field in $\text{Kill}_0(\widetilde{M}, x, \mathcal{F})$ leaves invariant the normal bundle, thence the map λ_x induces the following homomorphism of Lie algebras:

$$\lambda_x^\perp : \text{Kill}_0(\widetilde{M}, x, \mathcal{F}) \rightarrow \mathfrak{so}(T_x \mathcal{F}^\perp), \quad X \mapsto \lambda_x(X)|_{T_x \mathcal{F}^\perp}.$$

Observe that both $\rho_x(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ and $\mathcal{H}_0(x)$ lie inside of $\text{Kill}_0(\widetilde{M}, x, \mathcal{F})$.

Claim 1: λ_x^\perp is injective when it is restricted to $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))(x)$. By our choice of the element $x \in S$, the proof of this claim is similar to the proof of Lemma 2.1.

Claim 2: λ_x^\perp is injective when it is restricted to $\mathcal{H}_0(x)$. Recall that pseudo-Riemannian metrics are 1-rigid (see [2]). Therefore, a Killing vector field is completely determined by its 1-jet at x . Let $Z \in \mathcal{H}_0(x)$, then $\text{ev}_x(Z) = Z_x = 0$, so it is determined by its values $[Z, V]_x$ for V vector field on a neighborhood of x . Since $Z \in \mathcal{H}$ then $[Z, X^*]_x = 0$ for all $X \in (\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ in this way $[Z, V]_x = 0$ when $V_x \in T_x \mathcal{F}$. Hence, if $[Z, X]_x = 0$ when $V_x \in T_x \mathcal{F}^\perp$ this implies that $Z = 0$. Thence, we have that λ_x^\perp is injective when it is restricted to $\mathcal{H}_0(x)$.

Let $X \in (\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ and $Y \in \mathcal{H}_0(x)$, then

$$\begin{aligned} \lambda_x^\perp(X \cdot Y) &= \lambda_x^\perp([\widehat{\rho}_x(X), Y]) = \lambda_x^\perp([\rho_x(X) + X^*, Y]) \\ &= \lambda_x^\perp([\rho_x(X) + X^*, Y]) = [\lambda_x^\perp(\rho_x(X)), \lambda_x^\perp(Y)] \\ &= X \cdot \lambda_x^\perp(Y), \end{aligned}$$

which shows that the map λ_x^\perp restricted to $\mathcal{H}_0(x)$ is a homomorphism of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules. \square

It follows from the proof of Lemma 2.2 that $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a Lie subalgebra which contains $\mathcal{H}_0(x)$ as an ideal. We also have that $T_x \mathcal{F}$ is a trivial $\mathcal{H}_0(x)$ -module, therefore, by Lemma 1.5, $T_x \mathcal{F}^\perp$ is a $\mathcal{H}_0(x)$ -module which is non-trivial if and only if $\mathcal{H}_0(x)$ is non-trivial.

Remark 2.3. Let $x \in S$ as in the previous lemma, if $X \in \mathfrak{g}$ and $u \in T_x \widetilde{M}$ then, by Lemma 1.11, there exists $U \in \mathcal{H}$ such that $U_x = u$, hence

$$X \cdot u = [\rho_x(X), U]_x = [\rho_x(X) + X^*, U]_x = [\widehat{\rho}_x(X), U]_x.$$

In particular, we can define an action of $\mathcal{G}(x)$ on $T_x \mathcal{F}^\perp$ as following

$$(2.1) \quad \widehat{\rho}_x(X) \cdot u := [\widehat{\rho}_x(X), U]_x = [\rho_x(X), U]_x.$$

Let $x \in S$ be as in Lemma 2.2. If $X_1 \in \mathfrak{g}_1$, $u \in T_x \mathcal{F}_2$ and $v \in T_x \mathcal{F}^\perp$ then there exist $X_2 \in \mathfrak{g}_2$ and $V \in \mathcal{W}(x)$ such that $(X_2^*)_x = \widehat{\rho}_x(X_2) = u$ and $V_x = v$. By the proof of Lemma 1.8 we have that $X_1 \cdot \Omega_x^1(u \wedge v) = \Omega_x^1(X_1 \cdot (u \wedge v)) = 0$, since $[\widehat{\rho}_x(X), V] \in \mathcal{W}(x)$ for every $X \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Therefore, by the weak irreducibility of \widetilde{M} , we have that $\Omega_x^1 = d\omega_x^1|_{\wedge^2 T_x \mathcal{F}^\perp} \neq 0$. In a similar way we have that $\Omega_x^2 = d\omega_x^2|_{\wedge^2 T_x \mathcal{F}^\perp} \neq 0$. With the previous result we have that the 2-form $\Omega_x = d\omega_x|_{\wedge^2 T_x \mathcal{F}^\perp} : \wedge^2 T_x \mathcal{F}^\perp \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is surjective.

On the other hand, Lemma A.5 in [9] shows the existence of an isomorphism of $\mathfrak{so}(T_x \mathcal{F}^\perp)$ -modules $\varphi_x : \wedge^2 T_x \mathcal{F}^\perp \rightarrow \mathfrak{so}(T_x \mathcal{F}^\perp)$. Therefore, we will denote the linear map given by the composition $\Omega_x \circ \varphi_x^{-1} : \mathfrak{so}(T_x \mathcal{F}^\perp) \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the same symbol Ω_x .

Proposition 2.4. *For G and M as in Proposition 1.2. If $T\widetilde{M} = T\mathcal{F} \oplus T\mathcal{F}^\perp$ then for almost every $x \in S$, the following properties hold:*

- (1) *For every $X \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $Y \in \mathfrak{X}(\widetilde{M})$ we have*

$$\omega_x([\rho_x(X), Y]_x) = [X, \omega_x(Y)].$$

- (2) *The linear map $\Omega_x : \wedge^2 T_x \mathcal{F}^\perp \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2$ intertwines the homomorphism of Lie algebras $\widetilde{\rho}_x : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathcal{G}(x)$ for the actions of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and of $\mathcal{G}(x)$ on $T_x \mathcal{F}^\perp$ via (2.1). More precisely we have*

$$[X, \Omega_x(u \wedge v)] = \Omega_x[\widetilde{\rho}_x(X)(u \wedge v)]$$

for every $X \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $u, v \in T_x \mathcal{F}^\perp$.

- (3) *The linear map $\Omega_x : \mathfrak{so}(T_x \mathcal{F}^\perp) \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is $\mathcal{H}_0(x)$ via λ_x^\perp . More precisely, we have*

$$[\lambda_x^\perp(\mathcal{H}_0(x)), \mathfrak{so}(T_x \mathcal{F}^\perp)] \subset \ker(\Omega_x).$$

Proof. The proof is similar to the proof of the Proposition 3.10 in [9]. In that proof the authors prove this is true for $\mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$, where \mathfrak{g} is a simple Lie algebra.

Those arguments are the same in our case $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where \mathfrak{g}_1 and \mathfrak{g}_2 are simple Lie algebras and, therefore, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. \square

Remark 2.5. By Lemma 2.2 we have that $\mathfrak{so}(T_x \mathcal{F}^\perp) \simeq \mathfrak{so}(2n, 2n)$. On the other hand, by the decomposition of $\mathfrak{so}(2n, 2n)$ as a direct sum of irreducible $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules and the results of Proposition 2.4 we have that $\lambda_x^\perp(\mathcal{H}_0(x)) = 0$ and, therefore, $\mathcal{H}_0(x) = 0$.

Let $x \in S$ as in Lemma 2.2, by the previous remark, Lemma 1.11 and (2.1) the evaluation map

$$(2.2) \quad \mathrm{ev}_x : \mathcal{H} = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x) \oplus \mathcal{W}(x) \rightarrow T_x \mathcal{F}_1 \oplus T_x \mathcal{F}_2 \oplus T_x \mathcal{F}^\perp = T_x \widetilde{M}$$

is an isomorphism of $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ -modules.

Lemma 2.6. *For the $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ -action on M as in Proposition 1.2, assume that $n \geq 3$. For almost every $x \in S$ we have that \mathcal{H} is a simple Lie algebra isomorphic to $\mathfrak{sp}(n+1, \mathbb{R})$.*

Proof. We choose an element $x \in S$ which satisfies Lemma 2.2 and Proposition 2.4. By Remark 2.5 we have that $\mathcal{H} = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x) \oplus \mathcal{W}(x)$ is a Lie algebra where $\mathcal{G}(x) = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x)$, $\mathcal{G}_1(x) = \widehat{\rho}_x(\mathfrak{sp}(n, \mathbb{R}))$ and $\mathcal{G}_2(x) = \widehat{\rho}_x(\mathfrak{sp}(1, \mathbb{R}))$.

Since $\mathcal{G}(x) = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x)$ is a semisimple Lie subalgebra of \mathcal{H} isomorphic to $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$. Let \mathfrak{s} be a Levi factor of \mathcal{H} which contains to $\mathcal{G}(x)$.

Recall that the structure of \mathcal{H} as an $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module is given by the subalgebra $\mathcal{G}(x)$ and the Lie brackets in \mathcal{H} . Hence, since $\mathcal{G}(x) \subset \mathfrak{s}$ we have that \mathfrak{s} has a decomposition into $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules. Let \mathcal{U} be an $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -submodule of \mathcal{H} such that $\mathfrak{s} = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x) \oplus \mathcal{U}$. Therefore, we have a decomposition of \mathcal{H} into $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module as

$$\begin{aligned} \mathcal{H} &= \mathfrak{s} \oplus \text{rad}(\mathcal{H}) \\ &= \mathcal{G}(x) \oplus \mathcal{U} \oplus \text{rad}(\mathcal{H}) \\ &= \mathcal{G}_1(x) \oplus \mathcal{G}_2(x) \oplus \mathcal{U} \oplus \text{rad}(\mathcal{H}) \end{aligned}$$

that we compare with the decomposition of \mathcal{H} into $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module given by Lemma 2.2 and Remark 2.3, which is

$$\mathcal{H} = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x) \oplus \mathcal{W}(x).$$

Comparing the previous decomposition of \mathcal{H} as $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules we have two possibilities:

- (1) $\text{rad}(\mathcal{H}) = \mathcal{W}(x)$ and $\mathfrak{s} = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x)$.
- (2) \mathcal{H} is a semisimple Lie algebra.

Let us consider the case $\text{rad}(\mathcal{H}) = \mathcal{W}(x)$ and $\mathfrak{s} = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x)$.

Since $\text{rad}(\mathcal{H})$ is an ideal of \mathcal{H} we have that $[\mathcal{W}(x), \mathcal{W}(x)] \subset \mathcal{W}(x)$, therefore $\Omega_x^i([\mathcal{W}(x), \mathcal{W}(x)]) = 0$, for $i = 1, 2$. By the proof of Lemma 1.8 and (2.2) we have that this is not possible. Then case (1) is not possible.

Now assume that \mathcal{H} is a semisimple Lie algebra.

By properties of (2.2) and the action of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ into $T_x \mathcal{F}^\perp$ we have that $[\mathcal{G}_i(x), \mathcal{W}(x)] = \mathcal{W}(x)$, for $i = 1, 2$. Hence, if \mathfrak{h} is an ideal of \mathcal{H} containing $\mathcal{G}_i(x)$ then \mathfrak{h} must contain $\mathcal{W}(x)$. Therefore, we have that \mathcal{H} is a simple Lie algebra.

Since $\mathcal{W}(x) \simeq T_x \mathcal{F}^\perp$ as $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules we have then that $\wedge^2 \mathcal{W}(x)$ is isomorphic to $\wedge^2 T_x \mathcal{F}^\perp$ as $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules. On the other hand, by Lemma A.5 in [9] there is an isomorphism $\varphi : \wedge^2 T_x \mathcal{F}^\perp \rightarrow \mathfrak{so}(T_x \mathcal{F}^\perp)$ of $\mathfrak{so}(T_x \mathcal{F}^\perp)$ -modules.

From the decomposition of $\mathfrak{so}(T_x \mathcal{F}^\perp)$ (which is isomorphic to $\mathfrak{so}(2n, 2n)$) as a direct sum of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules (see the proof of Lemma A.2) and the fact that $[\mathcal{W}(x), \mathcal{W}(x)]$ has non-zero projection on $\mathcal{G}_1(x)$ and $\mathcal{G}_2(x)$ we have then that $[\mathcal{W}(x), \mathcal{W}(x)] = \mathcal{G}_1(x) \oplus \mathcal{G}_2(x)$. It follows in particular that $(\mathcal{H}, \mathcal{G}_1(x) \oplus \mathcal{G}_2(x))$ is a symmetric pair. Therefore, by Table II in [1], \mathcal{H} is isomorphic to $\mathfrak{sp}(n+1, \mathbb{R})$. \square

3. PROOF OF THE MAIN THEOREMS

In this section we assume the case $G = (\widetilde{\text{Sp}}(n, \mathbb{R}) \times \widetilde{\text{Sp}}(1, \mathbb{R}))$, for $n \geq 3$, which acts analytically and isometrically on a connected, analytic, finite-volume, complete, pseudo-Riemannian manifold M with a dense orbit, such that no factors of G acts trivially. Therefore, the results of Section 1 can apply to this case. We also assume that $\dim(M) \leq \dim(\widetilde{\text{Sp}}(n, \mathbb{R}) \times \widetilde{\text{Sp}}(1, \mathbb{R})) + 4n$.

Given the assumptions in the previous paragraph, by Lemma 1.5, we have the direct sum $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$. Here, we also assume that the manifold M is weakly irreducible.

By results in Section 2 we have the existence of a conull subset of \widetilde{M} which, we denote with the same letter S , such that every element $x \in S$ satisfies Lemmas 2.2 and 2.6. From now on we assume $x_0 \in S$.

Lemma 3.1. *There is an isomorphism*

$$\psi : \mathfrak{sp}(n+1, \mathbb{R}) = \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}) \oplus \mathbb{R}^{2n, 2n} \rightarrow \mathcal{G}_1(x_0) \oplus \mathcal{G}_2(x_0) \oplus \mathcal{W}(x) = \mathcal{H}$$

of Lie algebras that preserves the summands in that order. In particular, ψ is an isomorphism of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules.

Proof. Recall that $x_0 \in S$ satisfies Lemma 2.6, therefore, there is an algebra isomorphism $\psi_0 : \mathfrak{sp}(n+1, \mathbb{R}) \rightarrow \mathcal{H}$. The inverse image of $\mathcal{G}_1(x_0) \oplus \mathcal{G}_2(x_0)$ under the isomorphism ψ_0 induces the decomposition of $\mathfrak{sp}(n+1, \mathbb{R})$ as a direct sum of irreducible $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules, such decomposition satisfies that ψ_0 is an isomorphism of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules. \square

Now, let us fix an isomorphism of Lie algebras $\psi : \mathfrak{sp}(n+1, \mathbb{R}) \rightarrow \mathcal{H}$ as in Lemma 3.1. By Lemma 1.12, there is an analytic, isometric right $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ -action on \widetilde{M} . Hence, we can consider the next map:

$$f^\psi : \widetilde{\mathrm{Sp}}(n+1, \mathbb{R}) \rightarrow \widetilde{M}, \quad g \mapsto x_0 \cdot g$$

which satisfies $df_e^\psi(X) = X_{x_0}^* = \psi(X)_{x_0}$ for every $X \in \mathfrak{sp}(n+1, \mathbb{R})$. By properties of the map ψ and Lemma 2.2 we have that df_e^ψ is an isomorphism that maps $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ onto $T_{x_0}\mathcal{F}$ and $\mathbb{R}^{2n, 2n}$ onto $T_{x_0}\mathcal{F}^\perp$. The analyticity local diffeomorphism of f^ψ follows of the $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ -equivariance on its domain.

With a similar analysis to Lemma 3.2 in [8] we have our following result.

Lemma 3.2. *Let \widehat{g} be the metric on $\mathfrak{sp}(n+1, \mathbb{R})$ defined as the pullback under df^ψ of the metric g_{x_0} on $T_{x_0}\widetilde{M}$. Then, \widehat{g} is $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -invariant.*

By the previous Lemma and the results in Lemma A.10 we can rescale the metric along the bundles $T\mathcal{F}_1$, $T\mathcal{F}_2$ and $T\mathcal{F}^\perp$ in M such that the new metric \widetilde{g} on \widetilde{M} satisfies $(df^\psi)^*(\widetilde{g}_{x_0}) = K_{n+1}$, the Killing form of $\mathfrak{sp}(n+1, \mathbb{R})$.

Since the elements in \mathcal{H} preserve the decomposition TM as its direct sum $TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2 \oplus T\mathcal{F}^\perp$ then $\mathcal{H} \subset \mathrm{Kill}(\widetilde{M}, \widetilde{g})$. Hence, the elements of \mathcal{H} are Killing vector fields for the metric \widetilde{g} , therefore \widetilde{g} is invariant under the right $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ -action.

In a similar way we can observe that the isometric $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ -action on \widetilde{M} preserves our rescaled metric \widetilde{g} . We also note that the metric \widetilde{g} is the lift of a correspondingly metric \widetilde{g} in M .

Considering the bi-invariant metric on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ induced by the Killing form K_{n+1} , which we denote with the same symbol. The previous paragraphs show that the local diffeomorphism $f^\psi : (\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}), K) \rightarrow (\widetilde{M}, \widetilde{g})$ is a local isometry. Therefore, we have that f^ψ is an *isometry*, such result follows from Corollary 29 in [6, p. 202], the simply connectedness of \widetilde{M} and the completeness of $(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}), K)$.

Hence, by the previous remarks, we have the following result.

Lemma 3.3. *Let M and $G = (\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ as in Theorem A, then there is an analytic diffeomorphism $f : \widetilde{\mathrm{Sp}}(n+1, \mathbb{R}) \rightarrow \widetilde{M}$ and an analytic isometric right $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ on \widetilde{M} such that:*

- (1) *On \widetilde{M} , the right $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ -action and the left $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ -action commute with each other;*
- (2) *the map f is $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ -equivariant for the natural right $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ -action on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$;*

- (3) with the metric \tilde{g} , obtained by rescaling the original metric (g on M) on the summands of the direct decomposition $T\tilde{M} = T\mathcal{F}_1 \oplus T\mathcal{F}_2 \oplus T\mathcal{F}^\perp$, the map $f : (\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}), K) \rightarrow (\tilde{M}, \tilde{g})$ is an isometry, where K is the metric on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ induced by the Killing form of its Lie algebra.

First, by the results in [4] we have that $\mathrm{Iso}(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}))$ has finite many components and that $\mathrm{Iso}_0(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})) = L(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}))R(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}))$, where $L(g)$ (resp. $R(g)$) is the left (resp. right) translation map on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ by the element $g \in \widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$.

Let $\varrho : \widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}) \rightarrow \mathrm{Iso}_0(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}))$ be the homomorphism generated by the left action of $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$. By the previous paragraph, there are two homomorphism $\varrho_L, \varrho_R : \widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}) \rightarrow \widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ such that $\varrho(h) = L(\varrho_L(h))R(\varrho_R(h))$ for every $h \in (\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$.

Since the right $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ -action and the left action of $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ commute each other then $\varrho_R(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ lies in the center of $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$, such property and the fact that $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ is connected imply that $\varrho_R(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})) = e$. Therefore $\varrho = L \circ \varrho_L$.

On the other hand, by Lemma 3.3 and the previous paragraphs we have that the subgroup $\Sigma_0 = \pi_1(M) \cap \mathrm{Iso}_0(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}))$ (since $\pi_1(M) \subset \mathrm{Iso}(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R}))$) has finite index in $\pi_1(M)$. Considering that the action of $(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ is the lift of an action on M we have that the elements in Σ_0 commute with the elements of $\varrho(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$. Therefore, since $\sigma_0 = L(\sigma_1)R(\sigma_2)$ for some $\sigma_1, \sigma_2 \in \Sigma$ then $L(\sigma_1) \circ L(\varrho_L(h)) = L(\varrho_L(h)) \circ L(\sigma_1)$ for every $h \in (\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$, hence $\Sigma_0 \subset L(Z_{\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})}(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})))R(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$.

By the results in Lemma A.9 we have that $R(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ has finite index in $L(Z_{\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})}(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})))R(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$. In particular, $\Sigma = \Sigma_0 \cap R(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ is a finite index subgroup of Σ_0 , and therefore it has finite index in $\pi_1(M)$.

The natural identification of $R(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ with $\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})$ induces to consider Σ as a discrete subgroup of $\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})$ such that the quotient map $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})/\Sigma$ is a finite covering map of the manifold M . Let $\xi : \widetilde{\mathrm{Sp}}(n+1, \mathbb{R})/\Sigma \rightarrow M$ be the finite covering map, previously defined, for the left action of $\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})$ on $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})/\Sigma$ given by the homomorphism $\varrho_L : \widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}) \rightarrow \widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$, we have that the map ξ is $\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})$ -equivariant. We also observe that ξ is an isometry for the metric \tilde{g} , as it is defined in Lemma 3.2.

Finally, in order to complete the proof of Theorem A we only need to prove that the subgroup Σ is a lattice in $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$. Such result is shown in the following lemma which proof is similar to Lemma 3.4 in [8].

Lemma 3.4. *Let vol_g and $\mathrm{vol}_{\tilde{g}}$ define the volume elements on M , for the original metric and the rescaled metric \tilde{g} , in Lemma 3.2, respectively. Then, there is a constant $C_{\tilde{g}} > 0$ such that $\mathrm{vol}_{\tilde{g}} = C_{\tilde{g}}\mathrm{vol}_g$.*

APPENDIX A. MODULES AND REPRESENTATIONS

We start this appendix with the following result about decomposition into irreducible modules of non-compact simple Lie groups

Lemma A.1. *Let $G \subsetneq H$ be non-compact simple Lie groups and (π, V) an irreducible representation of H such that*

$$\pi|_G \simeq W \oplus \bigoplus_j W_j$$

is its direct sum decomposition into irreducible G -modules such that W has multiplicity 1 in V . If $\widetilde{W} \subset V$ is a G -invariant irreducible representation with $\widetilde{W} \simeq W$ then for every $g \in G$ we have that $\pi(g)(\widetilde{W}) = W$.

The previous lemma will be used to understand the inclusion of $\mathfrak{sp}(n, \mathbb{R})$ in $\mathfrak{so}(2n, 2n)$, for $n \geq 3$. In search of such understanding we recall the following:

$$(A.1) \quad \mathfrak{sp}(n, \mathbb{R}) = \{M \in \mathfrak{gl}_{2n}(\mathbb{R}) \mid M^T \widetilde{J} + \widetilde{J}M = 0\},$$

where

$$\widetilde{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

In a more explicitly way, we have

$$\mathfrak{sp}(n, \mathbb{R}) = \left\{ \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \in \mathfrak{gl}_{2n}(\mathbb{R}) \mid A \in \mathfrak{gl}_n(\mathbb{R}), B^T = B, C^T = C \right\}.$$

Note, by Table II of [1], that $(\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}))$ is a symmetric pair. Even more $\mathfrak{sl}(2n, \mathbb{R}) = \mathfrak{sp}(n, \mathbb{R}) \oplus \pi_2$ where π_2 is the irreducible representation of $\mathfrak{sp}(n, \mathbb{R})$ corresponding to its second highest weight ϖ_2 (see Theorem 5.5.15 and its immediate consequences in [3]).

On the other hand, we have that

$$(A.2) \quad \mathfrak{so}(2n, 2n) = \{M \in \mathfrak{gl}_{4n}(\mathbb{R}) \mid M^T J + JM = 0\},$$

where

$$J = \begin{bmatrix} 0 & I_{2n} \\ I_{2n} & 0 \end{bmatrix}.$$

That is

$$\mathfrak{so}(2n, 2n) = \left\{ \begin{bmatrix} A_0 & B_0 \\ C_0 & -A_0^T \end{bmatrix} \in \mathfrak{gl}_{4n}(\mathbb{R}) \mid A_0 \in \mathfrak{gl}_{2n}(\mathbb{R}), B_0, C_0 \in \mathfrak{so}(n, \mathbb{R}) \right\}.$$

Since $(\mathfrak{so}(2n, 2n), \mathfrak{sl}(2n, \mathbb{R}) \oplus \mathbb{R})$ is also a symmetric pair (see [1, Table II]) then we have a guarantee of an inclusion of $\mathfrak{sp}(n, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$. Therefore, an inclusion of the Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ into the Lie algebra $\mathfrak{so}(2n, 2n)$ is given in the following way

$$\begin{aligned} \mathfrak{sp}(n, \mathbb{R}) &\hookrightarrow \mathfrak{so}(2n, 2n) \\ \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} &\mapsto \begin{bmatrix} A & B & 0 & 0 \\ C & -A^T & 0 & 0 \\ 0 & 0 & -A^T & -C \\ 0 & 0 & -B & A \end{bmatrix}. \end{aligned}$$

Recall that $\mathfrak{gl}_{2n}(\mathbb{R}) = \mathfrak{sl}(2n, \mathbb{R}) \oplus \mathbb{R}$ then, by above, we have that $\mathfrak{gl}(2n, \mathbb{R}) = \mathfrak{sp}(n, \mathbb{R}) \oplus \pi_2 \oplus \mathbb{R}$. On the other hand, by Table II in [1], we have that

$$\mathfrak{so}(2n, 2n) = \mathfrak{sl}(2n, \mathbb{R}) \oplus \mathbb{R} \oplus \pi_2^1(\mathfrak{sl}(2n, \mathbb{R})) \oplus \pi_2^2(\mathfrak{sl}(2n, \mathbb{R}))$$

where $\pi_2^i(\mathfrak{sl}(2n, \mathbb{R}))$ denotes the irreducible representation of $\mathfrak{sl}(n, \mathbb{R})$ corresponding to its second highest weight, for $i = 1, 2$. Because $\pi_2^i(\mathfrak{sl}(2n, \mathbb{R})) = \pi_2 \oplus \mathbb{R}$, its decomposition as a direct sum of irreducible $\mathfrak{sp}(n, \mathbb{R})$ -modules, then

$$\mathfrak{so}(2n, 2n) \simeq \mathfrak{sp}(n, \mathbb{R}) \oplus \bigoplus_{i=1}^3 \pi_2^i \oplus \bigoplus_{i=1}^3 \pi_0^i,$$

as a direct sum of irreducible $\mathfrak{sp}(n, \mathbb{R})$ -modules where $\pi_0^i := \mathbb{R}$ is the trivial representation of $\mathfrak{sp}(n, \mathbb{R})$ corresponding to its highest weight ϖ_0 , for $i = 1, 2, 3$.

Let $W_0 \in \mathfrak{so}(2n, 2n)$ be an element which commutes with every element of the previous inclusion, of $\mathfrak{sp}(n, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$, then taking particular elements in $\mathfrak{sp}(n, \mathbb{R})$, it can be proven that

$$W_0 = \begin{bmatrix} aI_n & 0 & bI_n \\ 0 & aI_n & -bI_n \\ 0 & -cI_n & -aI_n \\ cI_n & 0 & 0 & -aI_n \end{bmatrix}$$

for some $a, b, c \in \mathbb{R}$. Therefore, with the above inclusion of $\mathfrak{sp}(n, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$ we have that an inclusion of $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$ is given as follow

$$\begin{aligned} \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}) &\hookrightarrow \mathfrak{so}(2n, 2n) \\ \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} + \begin{bmatrix} a & b \\ c & -a \end{bmatrix} &\mapsto \begin{bmatrix} A + aI_n & B & 0 & bI_n \\ C & -A^T + aI_n & -bI_n & 0 \\ 0 & -cI_n & -A^T - aI_n & -C \\ cI_n & 0 & -B & A - aI_n \end{bmatrix}. \end{aligned}$$

Lemma A.2. *There is, up to isomorphism, an unique inclusion of $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$.*

Proof. Since $(\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}))$ and $(\mathfrak{so}(2n, 2n), \mathfrak{sl}(2n, \mathbb{R}) \oplus \mathbb{R})$ are symmetric pairs we have an inclusion of $\mathfrak{sp}(n, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$. By the previous paragraphs we have that such inclusion induces a decomposition of $\mathfrak{so}(2n, 2n)$ into a direct sum of irreducible $\mathfrak{sp}(n, \mathbb{R})$ -modules as follow

$$\mathfrak{so}(2n, 2n) \simeq \mathfrak{sp}(n, \mathbb{R}) \oplus \bigoplus_{i=1}^3 \pi_2^i \oplus \bigoplus_{i=1}^3 \pi_0^i.$$

Lemma A.1 shows that the inclusion of $\mathfrak{sp}(n, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$, is unique up to isomorphism. On the other hand, by the simplicity of $\mathfrak{sp}(1, \mathbb{R})$ and since the inclusion of $\mathfrak{sp}(1, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$ is contained in $Z_{\mathfrak{so}(2n, 2n)}(\mathfrak{sp}(n, \mathbb{R}))$ (the centralizer of $\mathfrak{sp}(n, \mathbb{R})$ in $\mathfrak{so}(2n, 2n)$) then we have that the inclusion of $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$ is unique up to isomorphism. \square

An immediate consequence of the previous lemma is the following corollary.

Corollary A.3. *With the above inclusion of $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$, given in Lemma A.2, we have that $\mathbb{R}^{2n, 2n}$ is an irreducible $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module.*

Next, we analyze the representations of $\mathfrak{sp}(n, \mathbb{R})$ through the study of their correspondent complexification, all these facts can be found in [7].

Let \mathfrak{g}_0 be an real Lie algebra and let $\rho : \mathfrak{g}_0 \rightarrow \mathfrak{gl}(V_0)$ be a representation of \mathfrak{g}_0 in a real vector space V_0 . Let us denote $V = V_0(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$. Here, we have two complexification operations related to ρ . First, we have a complex representation $\rho^{\mathbb{C}} : \mathfrak{g}_0 \rightarrow \mathfrak{gl}(V)$, obtained extending any $\rho(x)$, $x \in \mathfrak{g}_0$, to a complex linear operator in V . Second, we can extend $\rho^{\mathbb{C}}$ to a homomorphism of complex Lie algebras $\rho(\mathbb{C}) : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$.

The following result uses the previous complex representations and it gives a classification of irreducible real representations.

Theorem A.4 ([7, Th 1, Sect. 8]). *Any irreducible real representation $\rho : \mathfrak{g}_0 \rightarrow \mathfrak{gl}(V_0)$ of a real Lie algebra \mathfrak{g}_0 satisfies precisely on of the following two conditions:*

- (i) $\rho^{\mathbb{C}}$ is an irreducible complex representation;
- (ii) $\rho = \rho'_{\mathbb{R}}$, where ρ' is an irreducible complex representation admitting no invariant real structures.

Conversely, any real representation ρ satisfying (i) or (ii) is irreducible.

Let $\rho : \mathfrak{g}_0 \rightarrow \mathfrak{gl}(V)$ be a self-conjugate irreducible complex representation. The *Cartan Index* of ρ is $\varepsilon(\rho) = \mathrm{sgn}(c) = \pm 1$, where c is defined by the following condition: $S^2 = ce$, where S is an automorphism of V commuting with ρ . By the results in Section 8 in [7], we have that an irreducible complex representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ admits an invariant real structure if and only if ρ is self-conjugate and its Cartan index is equal to 1.

By Theorem 3 in Section 8 and Table 5 in [7] we have that the irreducible complex representations of $\mathfrak{sp}(n, \mathbb{R})$ are self-conjugate and their Cartan index is always 1. Therefore, the study of real irreducible representations of $\mathfrak{sp}(n, \mathbb{R})$ is similar to the study of irreducible complex representations of $\mathfrak{sp}(n, \mathbb{C})$.

From Section 5.5.2 in [3] we have a bijection between (finite) complex representations of a complex semisimple Lie algebra \mathfrak{g} and the set of dominant integral weights associated to \mathfrak{g} . The dominant integral weights are of the form $n_1\varpi_1 + n_2\varpi_2 + \dots + n_k\varpi_k$ with $n_k \in \mathbb{N}$, where $\varpi_1, \varpi_2, \dots, \varpi_k$ are the fundamental weights of \mathfrak{g} .

It is clear that the dimension of the representation associated to $n_1\varpi_1 + n_2\varpi_2 + \dots + n_k\varpi_k$ is bigger or equal to the dimension of the representation associated to $n_j\varpi_j$ and this, if $n_j \neq 0$, to the dimension of the representation associated to ϖ_j , for every $j \in \{1, 2, \dots, k\}$.

In our case, $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$, and therefore for $\mathfrak{g}_0 = \mathfrak{sp}(n, \mathbb{R})$ we have that the fundamental weights are $\varpi_1, \varpi_2, \dots, \varpi_n$.

Lemma A.5. *The dimension of the representation of $\mathfrak{sp}(n, \mathbb{R})$ associated to ϖ_j is bigger than $4n$ when $n \geq 3$ and $2 \leq j \leq n$.*

Proof. By Corollary 5.5.17 in [3] we have that the dimension of the complex representation of $\mathfrak{sp}(n, \mathbb{C})$ associated to ϖ_j , and hence the real representation of $\mathfrak{sp}(n, \mathbb{R})$ associated to ϖ_j , is $\binom{2n}{j} - \binom{2n}{j-2}$ (with the convention that $\binom{m}{p} = 0$ when p is a negative integer).

If $j = 2$ then

$$\binom{2n}{2} - \binom{2n}{0} = n(2n-1) - 1,$$

which satisfies that

$$n(2n-1)-1 > 4n \iff j \geq 3.$$

If $j = 3$ then

$$\binom{2n}{3} - \binom{2n}{1} = \frac{2n(2n-1)(n-1)}{3} - 2n = \frac{2n(2n^2-3n-2)}{3}$$

satisfying that

$$\begin{aligned} \frac{2n(2n^2-3n-2)}{3} > 4n &\iff \frac{2n(2n^2-3n-8)}{3} > 0 \\ &\iff 2n(2n^2-3n-8) > 0 \\ &\iff 2n^2-3n-8 > 0 \\ &\iff n \geq 3. \end{aligned}$$

For $4 \leq k \leq n$ we have

$$\begin{aligned} \binom{2n}{k} - \binom{2n}{k-2} &= \binom{2n}{k-2} \left(\frac{(2n-k+3)(2n-k+4)}{(k-1)k} - 1 \right) \\ &\geq \binom{2n}{k-2} \left(\frac{(2k-k+3)(2k-k+4)}{(k-1)k} - 1 \right) \\ &= \binom{2n}{k-2} \left(\frac{(k+3)(k+4)}{(k-1)k} - 1 \right) \\ &= \binom{2n}{k-2} \frac{(k+3)(k+4) - k(k-1)}{(k-1)k} \\ &= \binom{2n}{k-2} \frac{8k+12}{(k-1)k} \\ &= \binom{2n}{k-2} \frac{8(k-1)+20}{(k-1)k} \\ &= \binom{2n}{k-2} \left(\frac{8}{k} + \frac{20}{(k-1)k} \right) \\ &> \binom{2n}{k-2} \frac{8}{k} \\ &\geq \binom{2n}{k-2} \frac{8}{n} \\ &\geq \binom{2n}{2} \frac{8}{n} \\ &= 8(2n-1), \end{aligned}$$

here

$$8(2n-1) > 4n \iff n > \frac{2}{3} \quad \text{in particular if } n \geq 4.$$

□

Remark A.6. First, by the definition of $\mathfrak{sp}(n, \mathbb{R})$, we have that the representation of $\mathfrak{sp}(n, \mathbb{R})$ on \mathbb{R}^{2n} , corresponding to its highest weight ϖ_1 , preserves a non-degenerate

skew-symmetric bilinear form. Hence, such representation cannot preserve a non-degenerate *symmetric* bilinear form. On the other hand, recall that the representation of $\mathfrak{sp}(n, \mathbb{R})$ on \mathbb{R} corresponds to the trivial homomorphism.

With the observations in Remark A.6 and Lemma A.5 we can now show a representation of $\mathfrak{sp}(n, \mathbb{R})$ with the minimal dimension preserving a non-degenerate symmetric bilinear form.

Lemma A.7. *The minimal dimension of a non-trivial representation of $\mathfrak{sp}(n, \mathbb{R})$ which preserves a non-degenerate symmetric bilinear form is $4n$, even more, such representation is isomorphic, as $\mathfrak{sp}(n, \mathbb{R})$ -module, to $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$.*

Proof. Let V be a non-trivial representation of $\mathfrak{sp}(n, \mathbb{R})$ which preserves a non-degenerate symmetric bilinear form. By Lemma A.5 and Remark A.6 we have that $\dim(V) = r \geq 2n$.

If $\dim(V) = 2n$ then, by Lemma A.5 and Remark A.6, we have that $V \simeq \mathbb{R}^{2n}$ (as $\mathfrak{sp}(n, \mathbb{R})$ -module) which can not preserve a non-degenerate symmetric bilinear form.

If $2n+1 \leq \dim(V) \leq 4n-1$ then, by Lemma A.5 and Remark A.6, we have that $V \simeq \mathbb{R}^{2n} \oplus \bigoplus_{j=1}^{r-2n} \mathbb{R}$. Let $\langle \cdot, \cdot \rangle$ be the non-degenerate symmetric bilinear form on $\mathbb{R}^{2n} \oplus \bigoplus_{j=1}^{r-2n} \mathbb{R}$ induced by its homomorphism with V , note that $\langle \cdot, \cdot \rangle$ is preserved by the action of $\mathfrak{sp}(n, \mathbb{R})$. Here, if $x \in \mathbb{R}^{2n}$ and $h \in \bigoplus_{j=1}^{r-2n} \mathbb{R}$ we have that for every $A \in \mathfrak{sp}(n, \mathbb{R})$

$$0 = \langle A \cdot x, h \rangle + \langle x, A \cdot h \rangle = \langle A \cdot x, h \rangle + \langle x, 0 \rangle = \langle A \cdot x, h \rangle.$$

Because the elements have been taken arbitrarily, that implies that $\langle \cdot, \cdot \rangle$ is non-degenerated when is restricted to \mathbb{R}^{2n} , which is not possible. Thence, resuming $\dim(V) \geq 4n$.

Since $(\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}))$ and $(\mathfrak{so}(2n, 2n), \mathfrak{sl}(2n, \mathbb{R}) \oplus \mathbb{R})$ are symmetric pairs then there is a non-trivial representation of $\mathfrak{sp}(n, \mathbb{R})$, with dimension $4n$, preserving a non-degenerate symmetric bilinear form.

If $\dim(V) = 4n$ then, by Lemma A.5, V must be isomorphic to $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$, $\mathbb{R}^{2n} \oplus \bigoplus_{j=1}^{2n} \mathbb{R}$ or well to $\bigoplus_{j=1}^{4n} \mathbb{R}$. The last two options are not possible as is shown previously. Therefore we have that V is isomorphic to $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. \square

And as consequence of the previous result and Corollary A.3 we have the next lemma.

Lemma A.8. *The decomposition of $\mathfrak{sp}(n+1, \mathbb{R})$ as a direct sum of irreducible $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules is given as $\mathfrak{sp}(n+1, \mathbb{R}) = \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}) \oplus \mathbb{R}^{2n, 2n}$.*

Proof. Recall, by [1], that $(\mathfrak{sp}(n+1, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ is a symmetric pair. On the other hand, since any Cartan involution on $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ can be extended to a Cartan involution on $\mathfrak{sp}(n+1, \mathbb{R})$ and all Cartan involution are conjugates we have then that the complement of $\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$ in $\mathfrak{sp}(n+1, \mathbb{R})$ is a non-degenerated vector subspace with dimension $4n$ which is a non-trivial $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module.

By Corollary A.3 and Lemma A.7 we have that such complement is isomorphic to $\mathbb{R}^{2n, 2n}$ as a $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module, which is irreducible. Therefore, we have our desired decomposition. \square

As a direct consequence of the previous lemma we know the centralizer of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ in $\mathfrak{sp}(n, \mathbb{R})$, such result can be found in the next lemma.

Lemma A.9. *Assume that $\rho : \widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}) \rightarrow \widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ is an homomorphism of Lie groups which is an immersion. Then, $Z_{\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})} \rho(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$, the centralizer of $\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R})$ in $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ contains the center of $\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})$ as a finite index subgroup.*

Proof. As a consequence of the decomposition of $\mathfrak{sp}(n+1, \mathbb{R})$ as a direct sum of irreducible $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -modules, shown in Lemma A.8, we have that $\mathfrak{z}_{\mathfrak{sp}(n+1, \mathbb{R})}(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})) = 0$, therefore $Z_{\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})} \rho(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ is discrete.

By Lemma 1.1.3.7 in [12] we have that $Z_{\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})} \rho(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$ is finite. Since $Z(\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})) \subseteq Z_{\widetilde{\mathrm{Sp}}(n+1, \mathbb{R})} \rho(\widetilde{\mathrm{Sp}}(n, \mathbb{R}) \times \widetilde{\mathrm{Sp}}(1, \mathbb{R}))$, therefore we have our result. \square

By the results in Lemma A.7 we have that $\mathbb{R}^{2n, 2n}$ is isomorphic to $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ as $\mathfrak{sp}(n, \mathbb{R})$ -module. On the other hand, by the inclusion of $\mathfrak{sp}(n, \mathbb{R})$ into $\mathfrak{so}(2n, 2n)$ (unique up to isomorphisms) as in Lemma A.2, and the remarks previous to such lemma, we have that the vector subspaces \mathbb{R}^{2n} belongs to the nullcone.

Next, we will see the properties of $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -invariants inner products on $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{sp}(n, \mathbb{R})$ and $\mathbb{R}^{2n, 2n}$.

Lemma A.10. *Let $\langle \cdot, \cdot \rangle_n$, $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_0$ be inner products on $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{sp}(1, \mathbb{R})$ and $\mathbb{R}^{2n, 2n}$, respectively. Assume that $\langle \cdot, \cdot \rangle_n$ is $\mathfrak{sp}(n, \mathbb{R})$ -invariant, $\langle \cdot, \cdot \rangle_1$ is $\mathfrak{sp}(1, \mathbb{R})$ -invariant and $\langle \cdot, \cdot \rangle_0$ is $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -invariant, for $n \geq 3$. Then there exist $a_0, a_1, a_n \in \mathbb{R}$ such that $a_0 \langle \cdot, \cdot \rangle_0 + a_1 \langle \cdot, \cdot \rangle_1 + a_n \langle \cdot, \cdot \rangle_n$ is the Killing form of $\mathfrak{sp}(n+1, \mathbb{R})$.*

Proof. The proof follows from Schur's Lemma, the irreducibility of $\mathbb{R}^{2n, 2n}$ as a $(\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R}))$ -module and the uniqueness of the Killing form of complex simple Lie algebras. \square

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